Influence of Dissipation on Quantum Tunneling in Macroscopic Systems

A. O. Caldeira and A. J. Leggett

School of Mathematical and Physical Sciences, University of Sussex, Brighton BN1 9QH, Sussex, United Kingdom
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A quantum system which can tunnel, at $T = 0$, out of a metastable state and whose interaction with its environment is adequately described in the classically accessible region by a phenomenological friction coefficient $\gamma$, is considered. By only assuming that the environment response is linear, it is found that dissipation multiplies the tunneling probability by the factor $\exp[-A\gamma(\Delta q)^2/\hbar]$, where $\Delta q$ is the "distance under the barrier" and $A$ is a numerical factor which is generally of order unity.

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One of the more intriguing prospects opened up by recent advances in understanding in the non-adiabatic regime in macroscopic systems and...
We confine ourselves to the case of zero temperature and consider a system characterized by a macroscopic coordinate \( q \) with which is associated a smooth potential-energy function \( V(q) \) with a metastable minimum; we choose the axes so that this lies at the origin \( (q = 0, V = 0) \). In what follows we denote the height of the barrier separating the metastable potential minimum from regions of lower potential by \( V_0 \) and its "width" [that is, the first nonzero value of \( q \) for which \( V(q) = 0 \)] by \( \Delta q \). The "mass" of the system is denoted by \( M \) and the frequency of small oscillations around the metastable equilibrium, \( M^{-1/2} \sqrt{V''(0)} \), by \( \omega_\circ \); we assume that \( \omega_\circ \ll V_0/\hbar \) and will work only to lowest order in the quantity \( \exp(-V_0/\hbar\omega_\circ) \). The system is assumed to be coupled to its environment in a way which is not necessarily known in detail, but which has the consequence that when the energy \( E \) satisfies the condition \( V_0 - E \gg \hbar\omega_\circ \) the expectation value of \( q(t) \) obeys, at least approximately, a classical equation of motion with friction coefficient \( \eta \), i.e.,

\[
\dot{q} + \eta \dot{q} = -dV/dq + F_{\text{ext}}(q),
\]

In particular, for \( E \ll V_0 \) and \( F_{\text{ext}} = 0 \) the system undergoes simple damped harmonic motion with

\[
\dot{x}_\alpha + \eta_\alpha \dot{x}_\alpha = -k_\alpha x_\alpha + \gamma_\alpha /\hbar \int dq_\beta K(q_i, q_f; x_\alpha; \tau)\exp(-E_\alpha/\hbar) \int dq_\alpha \psi_\alpha^*(q_i, x_\alpha) \psi_\alpha(q_f, x_\alpha) \exp(-E_\alpha/\hbar) dt/\hbar),
\]

where \( K(q_i, q_f; x_\alpha) \) is the quantum-mechanical transition amplitude for the "universe" (system plus environment) to go from coordinates \( (q_i, x_\alpha) \) at time zero to \( (q_f, x_\alpha) \) at time \( \tau \), and is given by the Feynman path integral

\[
K(q_i, q_f; x_\alpha) = \int dq_\alpha q_\alpha \psi_\alpha(q_i, x_\alpha) \int dq_\alpha(q_i) \alpha x_\alpha(q_i) \int dq_\alpha \psi_\alpha(q_f, x_\alpha) \exp(-L_E(q(t), x_\alpha(t)) dt/\hbar)
\]

and an inspection of the quantum probability \( K(q, q; \tau) \) for small \( q \) in the limit \( \tau \to 0 \) therefore gives both the probability density and the energy of the metastable ground state. In particular, the resulting \( L_E \) will have (after the appropriate analytic continuation procedures) a very small imaginary part which gives us the quantum tunneling rate.

To obtain a useful expression for \( K(q_i, q_f; \tau) \) we must exploit our assumption that the response of the environment is linear [at least for the amplitudes of \( q(t) \) important in the quantum tunneling process]. Since any system whose response is linear can be represented by a set of harmonic oscillators (and since by hypothesis the friction is linear in \( q \)), we may without loss of generality write the Euclidean Langrangian for the coupled system and environment in the form

\[
L_E = \frac{1}{2} M \dot{q}^2 + V(q) + \frac{1}{2} \sum_\alpha m_\alpha \dot{x}_\alpha^2 + \frac{1}{2} \sum_\alpha m_\alpha \omega_\alpha^2 x_\alpha^2 + q \sum_\alpha c_\alpha x_\alpha,
\]

where \( m_\alpha, \omega_\alpha, \) and \( c_\alpha \) are parameters which we do not need to know in detail (see below). The functional integrals over \( x_\alpha(q) \) in Eq. (3) and the integrals over \( x_\alpha(t) \) in Eq. (2) can now be (somewhat tedious-
ly) performed and yield, in the limit \( \tau \to \infty \),
\[
\tilde{R}(q_1, q_2; \tau) = \int_{q(0) = q_1}^{q(\tau) = q_2} Dq(\tau) \exp[-S_{\text{eff}}\{q(\tau)\}/\hbar],
\]
where the “effective action” \( S_{\text{eff}}\{q(\tau)\} \) is given by
\[
S_{\text{eff}}\{q(\tau)\} = \int_{0}^{\tau} \left[ \frac{1}{2} M \dot{q}^2 + V(q) \right] dt - \int_{-\infty}^{0} \int_{-\infty}^{\tau} \int_{-\infty}^{\tau} dt' dt'' \alpha(\tau - t') q(t') q(t'') + \text{const},
\]
\[
\alpha(\tau - t') = \sum_{a} c_2^{2/4M a_{\alpha}} \exp(-\omega_{\alpha}|t - t'|) = (1/2\pi) \int_{-\infty}^{+\infty} J(\omega) \exp(-\omega|t - t'|) d\omega \geq 0.
\]
In Eq. (7) \( q(\tau) \) is to be defined outside the region \( 0 < t < \tau \) by the prescription \( q(t + \tau) = q(\tau) \). (This is irrelevant to the semiclassical tunneling exponent but does affect other quantities.)

It is convenient to rewrite the second term in Eq. (7) with use of the identity \( q(t) q(t') = \frac{1}{2}[q(t^2) + q(t^2')] - \frac{1}{2}[q(t) - q(t')]^2 \). Then the squared terms can be lumped into \( V(q) \) and have the effect of shifting the small oscillation frequency \( \omega_0 \) downwards. Since this shift occurs in the classically allowed motion as well as in the quantum tunneling, it is unobservable and we shall simply imagine that it has been incorporated in the definition of \( V(q) \). Absorbing also the constant in (8) (the zero-point energy of the environment) into the zero of \( V(q) \), we see that the remaining correction to \( S_{\text{eff}} \) is always positive.

To proceed further we need to relate the quantity \( \alpha(\tau - t') \) defined in Eq. (8) to the phenomenological viscosity \( \eta \). We first note that since the characteristic times in the “bounce trajectory” (see below) are in general of order \( \omega_0^{-1} \), or longer, we need \( \alpha(\tau - t') \) only for times of this order, or equivalently \( J(\omega) \) for frequencies \( \lesssim \omega_0 \). Now, if the classical motion is to be determined by a well-defined friction coefficient at all (i.e., if the frictional force is to be proportional to the velocity), we must have in this frequency region the simple relation
\[
J(\omega \ll \omega_0) = \eta \omega.
\]
In the weak-damping limit this relation may be obtained simply by considering a large-amplitude classical motion of the system and equating the phenomenological expression for the power dissipated by it into the environment, \( \eta \omega^2 \), to the quantum-mechanical golden-rule expression written in terms of \( J \); in the more general case, it follows from a comparison of the ground-state probability distribution in the “harmonic” region \( V(q) \ll V_0 \) as calculated from Eq. (6) with the known expression for this quantity for a linear damped harmonic oscillator with friction coefficient \( \eta \). In view of Eqs. (8) and (9), we may, after the appropriate redefinition of the harmonic part of \( V(q) \) (see above), write the expression for \( S_{\text{eff}} \) in the following form which, if \( \omega_0 \) is the frequency at which \( J(\omega) \) deviates appreciably from its low-frequency form, is valid to lowest order in \( \omega_0/\omega_0 \):
set of the order of the undamped frequency $\omega_0$. The effects considered here are entirely associated with the dependence of the exponent $B$ on $\eta$, which is likely to be the overwhelmingly dominant effect in cases of practical interest.

In the limit of weak damping the correction $\Delta B$ to $B$ is obtained simply by evaluating the last term in Eq. (9) along the undamped trajectory $q(t) = (\Delta q)\gamma(t)$, where $f(t)$ is zero at $\pm \infty$ and reaches a maximum value of 1. This gives
\[
\Delta B = A_0 \gamma(\Delta q)^2,
\]
\[
A_0 = (1/4\pi) \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\xi [f(t) - f(t')]^2/(t - t')^2.
\]

For the practically important case of a potential of the form $\frac{1}{2} \omega_0^2 q^2 - \frac{1}{2} \lambda q^2$ we have $A_0 = (12/\pi^2)\xi(3)$. In the general case it is easy to see that $\Delta B$ can be written in the form $\eta(\Delta q)^2 \varphi(\alpha)$, where $\alpha \equiv \gamma/\omega_0$. There is space here only to quote without the proof that, at any rate for any potential which is bounded by expressions of the form $aq^2 - bq^4$, $\varphi(\alpha)$ is bounded above by a constant and below by a function of $\alpha$ which is in general of order unity and tends to zero for large $\alpha$ as $(\ln \alpha)^{-1}$. We strongly suspect that $\varphi(\alpha)$ actually tends to zero for large $\alpha$, if at all, even more slowly than logarithmically. At any rate the qualitative conclusion is clear: Linear friction suppresses quantum tunneling by a factor $\exp[\frac{1}{2} A(\Delta q)^2/H]$, where $A$ is in general of order unity.

To take our results over to the case of a SQUID described by the "resistively shunted junction" model, it is only necessary to replace $q$ in the above discussion by the trapped flux and $\eta$ by the experimentally measurable normal conductance of the junction (but cf. Ref. 12). We intend to discuss the details of this application elsewhere.

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5. W. den Boer and R. de Bruyn Ouboter, Physica (Utrecht) B/C 98, 185 (1980). These authors show that their results are compatible with the WKB predictions.
8. It is not necessary for the validity of our results to this order that the damping $\gamma$ be small compared to $\omega_0/H$.
12. In the case of a SQUID careful consideration of the physical meaning of the $x_q$ shows it is necessary to add to $E_F$ a term of the form $\zeta \omega_0^2/2m_0 \omega_0^2 \omega^2$ whose function is exactly to cancel the frequency shift discussed below (which is clearly unphysical in this case). This point will be discussed elsewhere.
14. B. Yurke and O. Yurke, to be published; cf., also Caldeira, Ref. 4.
15. Since in a bounce $q$ is exponentially small except for a time of order $\omega_0^{-1}$, the limits of integration in (9) may be extended from $-\infty$ to $+\infty$.\n
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